Notes on the stopping criteria in the W4 method

Hirotada Okawa

Waseda Institute for Advanced Study, Waseda University, Tokyo 169-0051, Japan

1. Sample problem

The structure inside non-rotating stars that are composed of gases can be described by the so-called Lane-Emden equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \theta^n = 0, \tag{1}$$

where r, θ, and n denote the radial coordinate, a non-dimensional density, and a polytropic index in the equation of state for the gas, respectively. The non-dimensional radius x is introduced as $x \equiv r/R$ where R is the surface radius of the star and $0 \le x \le 1$ and then the equation is rewritten as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}\theta}{\mathrm{d}x} + R^2\theta^n = 0. \tag{2}$$

The second-order discretization of this equation on the equally-spaced grid in x ($0 \le x \le 1$) gives the following nonlinear system of equations:

$$\frac{F_j}{(\Delta x)^2} \quad := \quad \frac{\theta_{j+1} - 2\theta_j + \theta_{j-1}}{(\Delta x)^2} + \frac{1}{x_j}\frac{\theta_{j+1} - \theta_{j-1}}{\Delta x} + R^2\theta_j^n = 0, \tag{3}$$

where j runs from 1 to N_s, the number of grid points; $\Delta x = 1/N_s$. The boundary conditions need to be imposed: (i) the density at the surface should vanish, $\theta_{N_s} = 0$ and (ii) the density at the center of the star is normalized and smooth, $\theta_0 = 1$, and $\frac{\mathrm{d}\theta}{\mathrm{d}x}\,|_{x=0} = \frac{\theta_1 - \theta_{-1}}{2\Delta x} = 0$.

The variables and equations to be solved are summarized as

$$\boldsymbol{x} \quad = \quad (\theta_1, \theta_2, \cdots, \theta_{N_s-1}, r_*)\,, \tag{4}$$

$$\boldsymbol{F} \quad = \quad (F_1, F_2, \cdots, F_{N_s-1}, F_0) = \mathbf{0}, \tag{5}$$

where r_* is defined by $r_* := R^2(\Delta x)^2$ for visibility and $F_0 = 6(\theta_1 - 1) + r_*$ by applying L'Hopital's rule to Eq. (2) at $x = 0$ and imposing the boundary condition. It is evident that the number of the variables is equal to that of the equations as it should. These equations with the boundary conditions are solved with the W4 method with the LH decomposition.

References

Email address: `h.okawa@aoni.waseda.jp` (Hirotada Okawa)

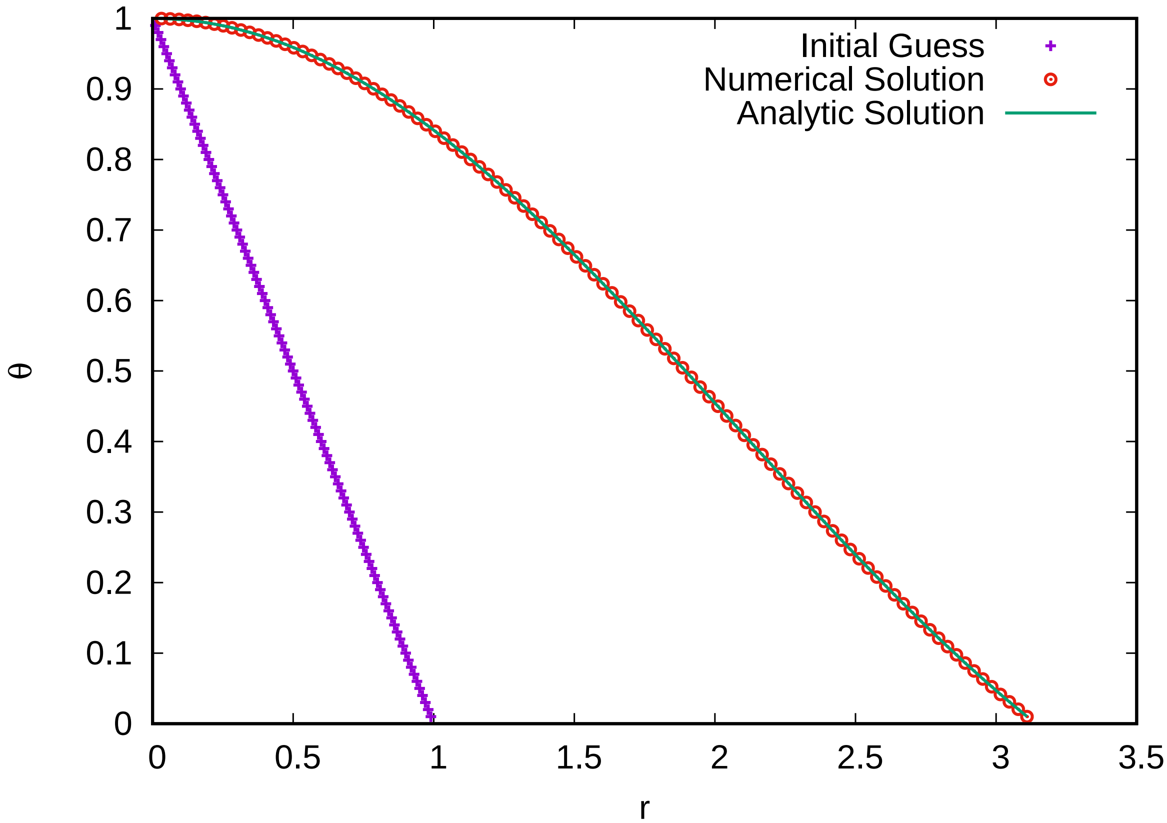

Figure 1: The solution of Lane-Emden equation with the parameter $n = 1$. The purple pluses show the initial guess and the red circles show the numerical solution obtained by the W4 method. The green solid curve denotes the analytic solution $\sin x/x$.

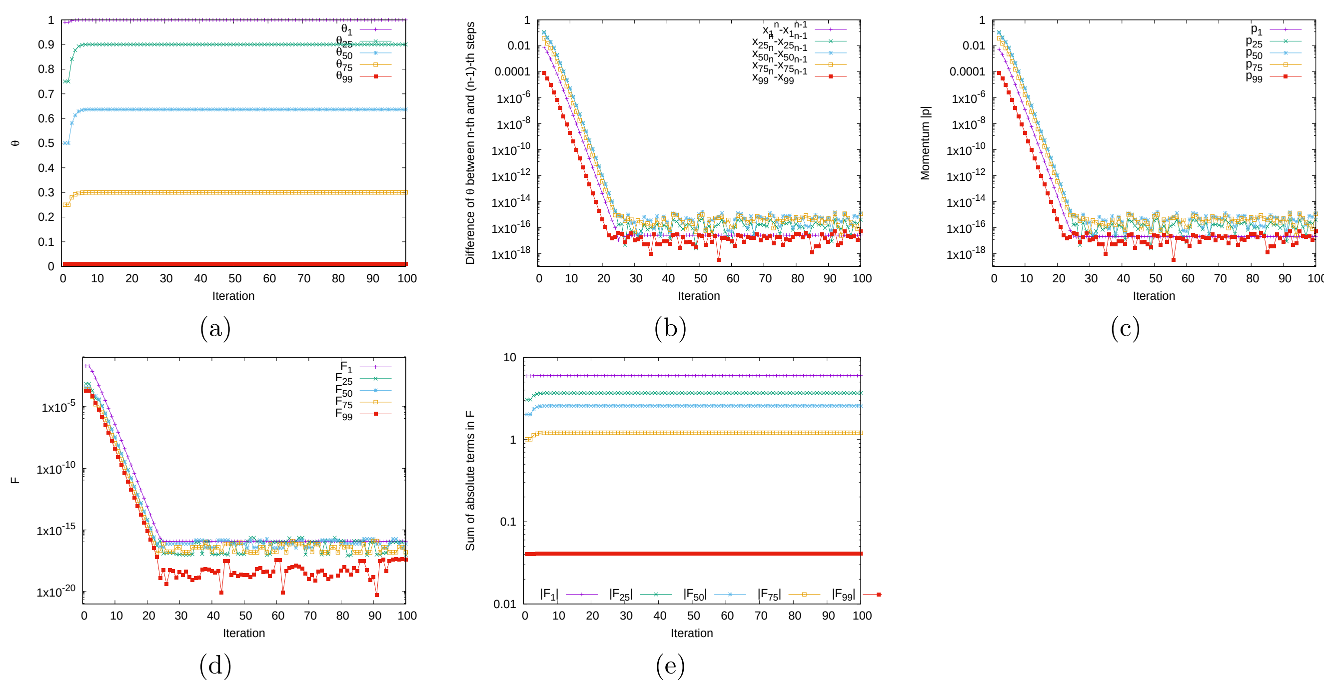

Figure 2: (a) Evolution of each θ during the iteration. (b) Difference between the present step value and previous step value of each θ. (c) Evolution of each momentum p_θ in the W4 method during the iteration. (d) Function F_j to be solved during the iteration. (e) Sum of all absolute terms in the function F_j.