How to numerically solve Differential Equations?

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Abstract Various phenomena in physics are described by (partial) **differential equations(DEs)**. However, it is known that **most differential equations cannot be solved analytically**. Even numerically, no method exists so far for general DEs, while I propose a new scheme utilizing **the W4 method** which is a recently proposed root-finding method. This poster tries to explain how differential equations are solved numerically.

1. Introduction

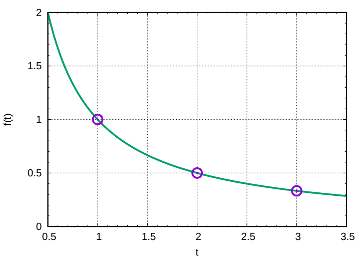
Let's solve a simple DE:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = -f(t)^2. \qquad (1)$$

The solution is

$$f(t) = \frac{1}{t}, \qquad (2)$$

since it satisfies Eq. (1), i.e., $\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{1}{t^2} = -\left(\frac{1}{t}\right)^2 = -f^2$.



Suppose t denotes time and f is some physical quantity like temperature, which obeys the differential equation (1). Then we will observe the values $f(t)$ at any time t as in the upper figure.

2. Riccati's differential equation

However, such a simple DE cannot be solved analytically if the following term $-ct$ is just added:

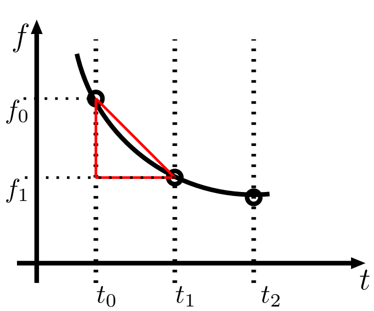
$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = -f(t)^2 - ct, \qquad (3)$$

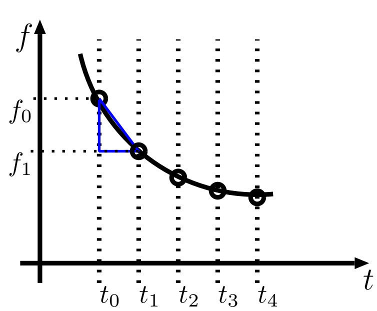
where c is some nonzero real number. It is obvious that $1/t$ is no longer the solution of Eq. (3). Even if we know the values $f(t=1)$ and $f(t=3)$, the solution during $1 < t < 3$ cannot be found analytically.

3. How to usually solve DEs?

Very roughly speaking, the derivative of functions means the slope of its increase (or decrese). This is the idea to discretize the continuous Eq. (3) into Eq. (4). When we use many points, the slope can be approximated well as in the lower figures.

$$\frac{f_{i+1} - f_i}{t_{i+1} - t_i} = -f_i^2 - ct_i, \qquad (4)$$





4. A set of nonlinear equations

The problem to solve a differential equation is to solve a set of equations, $(F_0, F_1, \cdots, F_{N-1})$, which must be satisfied at all discretized points.

$$\begin{aligned} F_0 &\equiv \frac{f_1 - f_0}{t_1 - t_0} + f_0^2 + ct_0 = 0, \\ F_1 &\equiv \frac{f_2 - f_1}{t_2 - t_1} + f_1^2 + ct_1 = 0, \\ &\vdots \\ F_{N-1} &\equiv \frac{f_N - f_{N-1}}{t_N - t_{N-1}} + f_{N-1}^2 + t_{N-1} = 0. \end{aligned}$$

Unknown variables: $(c, f_1, f_2, \cdots, f_{N-1})$

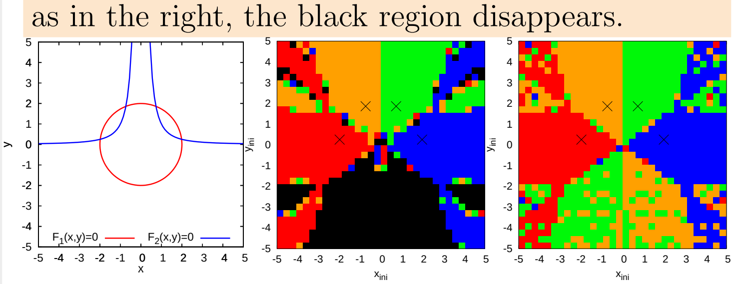
Boundary conditions: $f_0 = 1, f_N = 1/3$

(suppose the values at $t_0 = 1$ and $t_N = 3$ are observed.)

5. Root-finding method

An idea to solve such a set of nonlinear equations had been already proposed by Isaac Newton and Joseph Raphson in 17th century and is still the leading method. It is known, however, that we need finetune an initial guess to the solution for this iterative method, unfortunately.

We have proposed a new method, the W4 method, inspired by the mathematical property of damped oscillators. Below we apply it to the simpler problem to find the cross points between the circle and curve in the left figure. As shown in the center, the Newton-Raphson method can find the solution(Crosses) only when we start from an initial guess in the colored region, whereas it cannot find any solution from black region. By the W4 method as in the right, the black region disappears.



6. Solution to Riccati's DE

We applied the W4 method to Riccati's differential equation and successfully found the solution (Lower) from the initial (Upper).